Fokker-Planck equation in a wedge domain: Anomalous diffusion and survival probability

E. K. Lenzi,[1] L. R. Evangelista,[1] M. K. Lenzi,[2] and L. R. da Silva[3]
[1]*Departamento de Física, Universidade Estadual de Maringá, Av. Colombo 5790, 87020-900 Maringá, PR, Brazil*
[2]*Departamento de Engenharia Química, Universidade Federal do Paraná, Setor de Tecnologia–Jardim das Américas, Caixa Postal 19011, 81531-990 Curitiba, PR, Brazil*
[3]*Departamento de Física, Universidade Federal do Rio Grande do Norte, 59072-970 Natal, RN, Brazil*
(Received 12 April 2009; revised manuscript received 27 June 2009; published 28 August 2009)

We obtain exact solutions and the survival probability for a Fokker-Planck equation subjected to the two-dimensional wedge domain. We consider a spatial dependence in the diffusion coefficient and the presence of external forces. The results show an anomalous spreading of the solution and, consequently, a nonusual behavior of the survival probability which can be connected to anomalous diffusion.

DOI: 10.1103/PhysRevE.80.021131 PACS number(s): 05.40.Jc, 82.20.Db, 66.10.C−, 05.60.−k

I. INTRODUCTION

Diffusion is one of the most important phenomena in nature and is present in several fields of the physics. Typical situations are described in terms of a Fokker-Planck equation and characterized by a mean-square displacement that is asymptotically linear in time, i.e., $\langle (r-\langle r\rangle)^2\rangle \sim t$. These features are deeply related to the central limit theorem and the Markovian nature of this stochastic process. However, the large number of experimental observations show that more complex processes, in which the mean-square displacement is not proportional to t, also occur in nature. These situations can be found, for instance, in CTAB micelles [aggregates of amphiphillic molecules—cethyl trimethyl ammonium bromide (CTAB)] dissolved in salted water [1,2], the analysis of heartbeat histograms in a healthy individual [3], chaotic transport in laminar fluid flow of a water-glycerol mixture in a rapidly rotating annulus [4], subrecoil laser cooling [5], particle chaotic dynamics along the stochastic web associated with a $d=3$ Hamiltonian flow with hexagonal symmetry in a plane [6–8], conservative motion in a $d=2$ periodic potential [9], transport of fluid in porous media (see [10] and references therein), surface growth [10], and many other interesting physical systems.

Several formalisms have been employed to investigate these systems presenting anomalous diffusion. Some of them are based on extensions of the Fokker-Planck equation by incorporating fractional derivatives [11–23], nonlinear terms [24,25] or spatial, and time dependence in the diffusion coefficient [26–29]. They have been applied in a rich variety of scenarios such as systems with trapping or recombination [30], polymer translocation through a nanopore [31], anomalous transport in disordered systems [32], diffusion on fractals [33,34], and microporous materials [35], stochastic acceleration of particles in plasmas [36,37], turbulence [38], and modulation of electron transfer kinetics by protein conformational fluctuations [39]. The advantage of these equations is the simple way to deal with boundary values problems and to incorporate forced fields which, in other formalisms, may lead one to cumbersome situations. Having these scenarios in mind, we focus our attention to the solutions and the survival probability in a wedge domain (see Fig. 1) of the Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) = \nabla\cdot[\mathcal{D}(r)\nabla\rho(\mathbf{r},t)] - \nabla\cdot[\bar{F}(\mathbf{r},t)\rho(\mathbf{r},t)], \quad (1)$$

where $\mathbf{r}=(r,\theta)$, $\mathcal{D}(r)$ is the (dimensionless) diffusion coefficient given by $\mathcal{D}(r)=\mathcal{D}r^{-\eta}$ and $\bar{F}(\mathbf{r},t)$ is the (dimensionless) external force (*drift*) associated with the potential $V(\mathbf{r},t) = kr^2/2+(\mathcal{K}/\eta)(1/r^{\eta}-1)$. This potential can be considered as an extension of the logarithmic potential used, for instance, to establish the connection between the fractional diffusion coefficient and the generalized mobility [40], and the external force obtained from it has as particular case the Ornstein-Uhlenbeck [41] and the Rayleigh [42] processes. We underline that the solution of Eq. (1) in absence of external force, for free boundary condition $[\lim_{|\mathbf{r}|\to\infty}\rho(r,t)=0]$, is a stretched exponential, i.e., $\rho(\mathbf{r},t) \propto e^{-r^{2+\eta}/((2+\eta)^2\mathcal{D}t)}/t^{2/(2+\eta)}$, and consequently $\langle (r-\langle r\rangle)^2\rangle \propto t^{2/(2+\eta)}$ which can be related to a sub or superdiffusive process. We first consider a confined region, in contrast to the

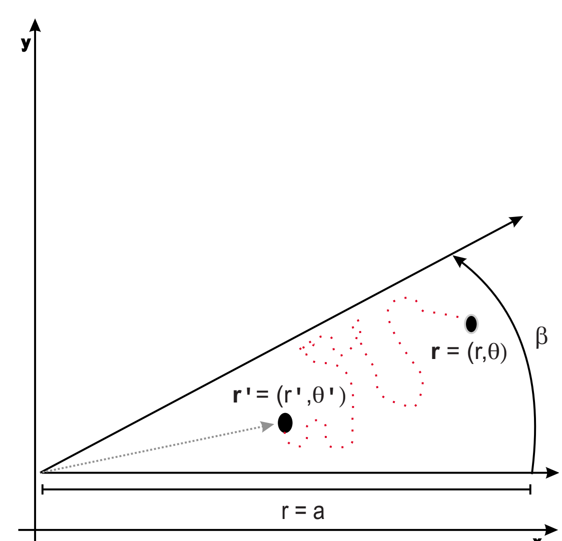

FIG. 1. (Color online) Geometry of the wedge domain (region) considered here. Note that similar structure with $a\to\infty$ was worked out in [43,44] for the usual diffusion equation with a constant diffusion coefficient in absence of external forces.

situation analyzed in [43,44], characterized by the boundary conditions $\lim_{r\to a}\rho(\mathbf{r},t)=\lim_{r\to 0}\rho(\mathbf{r},t)=0$ and $\lim_{\theta\to\beta}\rho(\mathbf{r},t)=\lim_{\theta\to 0}\rho(\mathbf{r},t)=0$, with the diffusion coefficient $\mathcal{D}(r)=\mathcal{D}r^{-\eta}$. For this case, we also analyze the mean first passage time distribution [44]. Afterwards, we extend the solution to $a\to\infty$ and incorporate the external force $\bar{F}(\mathbf{r},t)=-kr\hat{r}+\mathcal{K}/r^{1+\eta}\hat{r}$, analyzing the first passage time distribution. The results obtained for this geometry may find application in several physical situations such as flow of a viscous fluid [45], one-dimensional diffusion controlled reaction processes [44,46], random velocity field [47], and also make possible to investigate situations characterized by anomalous diffusion processes which are modeled by $\eta\neq 0$.

In these cases, we employ the initial condition $\rho(\mathbf{r},0)=\bar{\rho}(\mathbf{r})$ with $\bar{\rho}(\mathbf{r})$ normalized, i.e., $\int_0^\beta d\theta\int_0^\infty dr r\bar{\rho}(\mathbf{r})=1$. In this manner, we extend results found in [43,44] by including a spatial dependence in the diffusion coefficient and by considering the presence of an external force. These developments are performed in Sec. II while in Sec. III some conclusions are presented.

II. FOKKER-PLANCK EQUATION AND SURVIVAL PROBABILITY

We shall address this section to the discussion of the diffusion equation in a confined wedge domain characterized by the boundary conditions defined above. This discussion is accomplished by the diffusion coefficient $\mathcal{D}(r,t)=\mathcal{D}r^{-\eta}$, which has been used to investigate physical situations such as diffusion on fractals [33,34], turbulence [38], and fast electrons in a hot plasma in the presence of a electric field [48]. For this case, Eq. (1) is given by

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t)=\mathcal{D}\,\nabla\cdot[r^{-\eta}\nabla\rho(\mathbf{r},t)]. \tag{2}$$

In order to solve this equation, we use the Laplace transform and Green function approach [49]. Applying the Laplace transform to Eq. (2), one obtains

$$\mathcal{D}\,\nabla\cdot[r^{-\eta}\nabla\rho(\mathbf{r},s)]=s\rho(\mathbf{r},s)-\rho(\mathbf{r},0), \tag{3}$$

whose solution can be put in the form

$$\rho(\mathbf{r},s)=-\int_0^a dr'r'\int_0^\beta d\theta'\rho(\mathbf{r}',0)\mathcal{G}(\mathbf{r},\mathbf{r}',s), \tag{4}$$

with the Green function being governed by the equation:

$$\mathcal{D}\,\nabla\cdot[r^{-\eta}\nabla\mathcal{G}(\mathbf{r},\mathbf{r}',s)]-s\mathcal{G}(\mathbf{r},\mathbf{r}',s)=\delta(\mathbf{r}-\mathbf{r}'), \tag{5}$$

and subjected to the conditions $\lim_{r\to a}\mathcal{G}(\mathbf{r},\mathbf{r}',s)=\lim_{r\to 0}\mathcal{G}(\mathbf{r},\mathbf{r}',s)=0$ and $\lim_{\theta\to\beta}\mathcal{G}(\mathbf{r},\mathbf{r}',s)=\lim_{\theta\to 0}\mathcal{G}(\mathbf{r},\mathbf{r}',s)=0$. By using the eigenfunctions of the Sturm-Liouville problem related to the spatial operator of Eq. (5), i.e., $\nabla\cdot\{[r^{-\eta}\nabla\Psi(\mathbf{r},k_{mn})]\}=-k_{mn}^2\Psi(\mathbf{r},k_{mn})$ where $\Psi(\mathbf{r},k_{mn})$ satisfies the same boundary conditions of the Green function, it is possible to show that the Green function of Eq. (5) may be written as

$$\mathcal{G}(\mathbf{r},\mathbf{r}',s)=-\sum_{n,m=1}^{\infty}\Phi(k_{mn},s)\Psi(\mathbf{r},k_{mn})\Psi(\mathbf{r}',k_{mn}), \tag{6}$$

with

$$\Phi(k_{mn},s)=\mathcal{N}_{mn}/(s+\mathcal{D}k_{mn}^2), \tag{7}$$

which has as inverse Laplace transform $\Phi(k_{mn},t)=\mathcal{N}_{mn}\exp(-\mathcal{D}k_{mn}^2t)$. Notice that the result obtained for the Green function in Eq. (6) may be extended to fractional diffusion equations by replacing $\mathcal{D}$ with an arbitrary function $\overline{\mathcal{D}}(s)$. A typical example is $\mathcal{D}\to\mathcal{D}s^{1-\gamma}$ which, when substituted in Eq. (7), yields $\Phi(k_{mn},t)=\mathcal{N}_{mn}\mathrm{E}_\gamma(-\mathcal{D}k_{mn}^2t^\gamma)$, where $\mathrm{E}_\gamma(x)$ is the Mittag-Leffler function [50]. Thus, the changes produced by a fractional time derivative (or fractional time derivatives of distributed order) in Eq. (6) are manifested by $\Phi(k_{mn},t)$ which contains the time dependence of the solution. The eigenfunction, after solving the equation for $\Psi(\mathbf{r},k_{mn})$ by using the method of variable separation with suitable boundary conditions, is given by

$$\Psi(\mathbf{r},k_{mn})=r^{\eta/2}\mathrm{J}_{\nu_m}\left(\frac{2k_{mn}}{2+\eta}r^{1/2(2+\eta)}\right)\sin\left(\frac{m\pi}{\beta}\theta\right), \tag{8}$$

where $\eta\geq 0$, $\mathrm{J}_\nu(x)$ is the Bessel function, $\nu_m=[\eta/(2+\eta)]\sqrt{1+[2m\pi/(\beta\eta)]^2}$, and

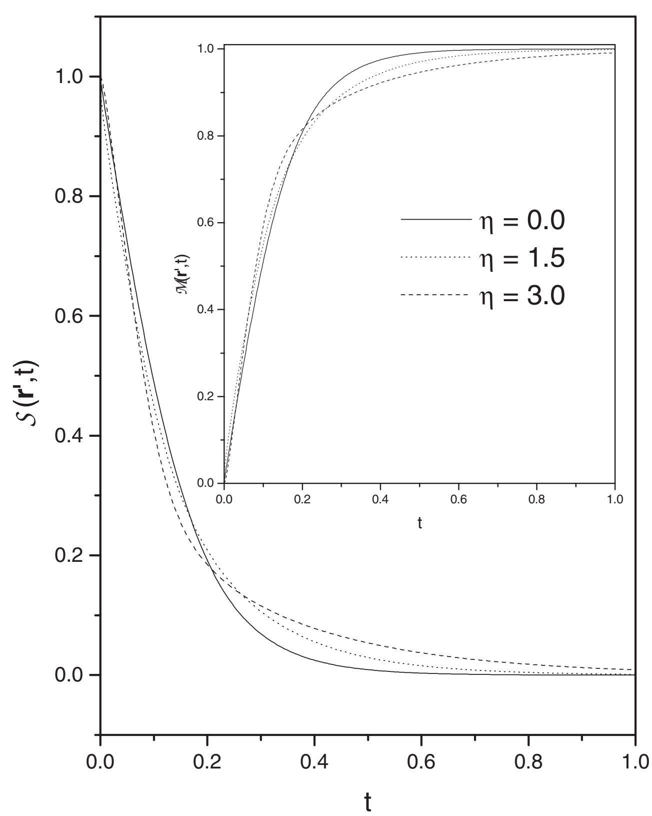

FIG. 2. Behavior of $\mathcal{S}(\mathbf{r}',t)$ versus t, which illustrates Eq. (10), for typical values of η by considering, for simplicity, $\mathcal{D}=1$, $a=2$, $\beta=\pi/3$ and the initial condition $\bar{\rho}(\bar{r})=1/r\delta(r-1)\delta(\theta-\pi/5)$. The inset figure is $\mathcal{M}(\mathbf{r}',t)$ versus t, which is useful to evidence the rate at which the particles are removed from the system in each case.

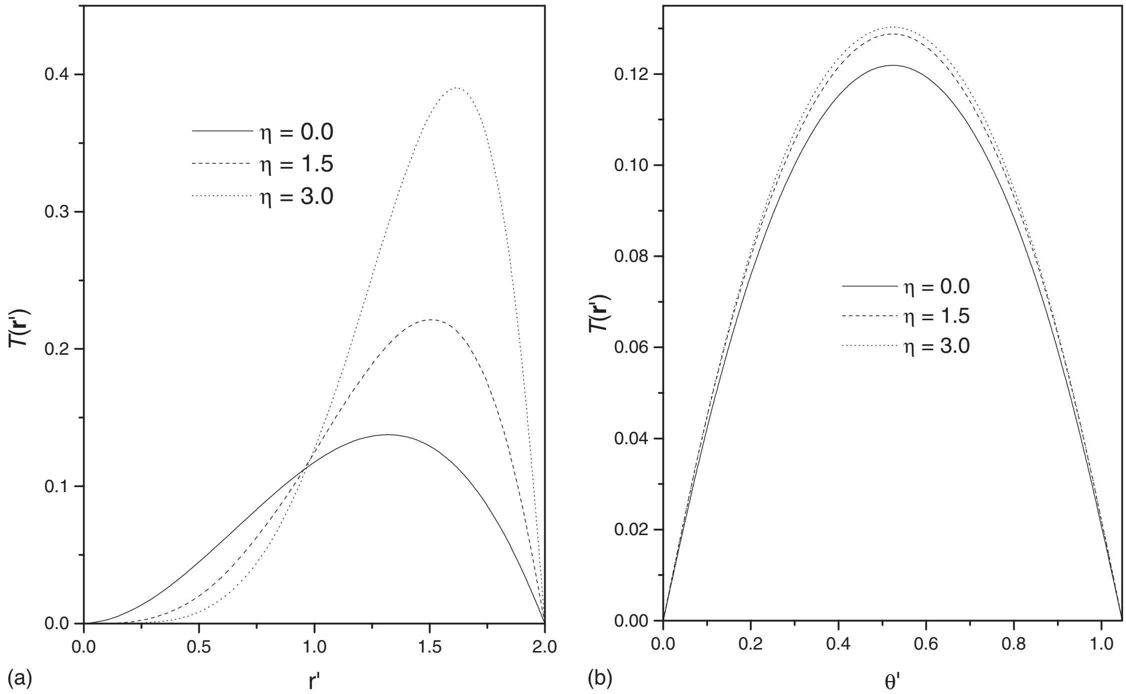

FIG. 3. Behavior of the mean first passage time obtained from Eq. (10) is illustrated in Figs. 3(a) and 3(b) for typical values of η. We consider, for simplicity, $\mathcal{D}=1$, $a=2$, $\beta=\pi/3$ and the initial condition $\bar{\rho}(\mathbf{r})=1/r\delta(r-1)\delta(\theta-\pi/5)$.

$$\mathcal{N}_{nm}=\frac{2(2+\eta)}{\beta a^{2+\eta}\mathrm{J}_{\nu_m+1}\left(\frac{2k_{mn}}{2+\eta}a^{1/2(2+\eta)}\right)}, \tag{9}$$

with the eigenvalues k_{mn} determined by solving the equation $\mathrm{J}_{\nu_m}(\frac{2k_{mn}}{2+\eta}a^{1/2(2+\eta)})=0$. To proceed, we investigate, for this process, the survival probability distribution which is related to the first passage time distribution. Using the previous results, it is possible to show that the survival probability distribution is given by

$$\mathcal{S}(\mathbf{r}',t)=\int_0^a dr r\int_0^\beta d\theta\rho(\mathbf{r},t)=\sum_{n,m=1}^{\infty}\mathcal{I}_{mn}\Phi(k_{mn},t)\Psi(\mathbf{r}',k_{mn}), \tag{10}$$

for the initial condition $\rho(\mathbf{r},0)=1/r\delta(r-r')\delta(\theta-\theta')$ with

$$\mathcal{I}_{mn}=\frac{2a\beta}{(2+\eta)m\pi}\frac{a^{1/2(2+\eta)}}{\mathcal{A}_{mn}^{2/2+\eta}}(1-(-1)^m)\times\left[\frac{2^{2/2+\eta}\Gamma\left(\frac{1}{2+\eta}+\frac{\nu_m+1}{2}\right)}{\mathcal{A}_{mn}\Gamma\left(\frac{\nu_m}{2}+\frac{\eta}{2(2+\eta)}\right)}+\left(\nu_m-\frac{\eta}{2+\eta}\right)\mathrm{J}_{\nu_m}(\mathcal{A}_{mn})\times\mathrm{S}_{-\eta/2+\eta,\nu_m-1}(\mathcal{A}_{mn})-\mathrm{J}_{\nu_m-1}(\mathcal{A}_{mn})\mathrm{S}_{2/2+\eta,\nu_m}(\mathcal{A}_{mn})\right], \tag{11}$$

where $\mathcal{A}_{mn}=2k_{mn}a^{1/2(2+\eta)}/(2+\eta)$, $\mathrm{S}_{\alpha,\gamma}(x)$ is the Lommel function [51] and $\bar{\rho}(\bar{r})$ represents how the system is initially distributed. Figure 2 shows the behavior of the survival probability obtained from Eq. (10) for typical values of η. Note that for $\eta>0$ we have more particles absorbed, i.e., removed from the system, by the surfaces than in the usual case, $\eta=0$, for small time. However, a small quantity of particles spend more time than in the usual case ($\eta=0$) to be absorbed when $\eta>0$. The last statement may be verified by analyzing the asymptotic behavior of Eq. (10) present in Fig. 2. In particular, it may be obtained from Eq. (10) by expanding the series and keeping the initial terms, i.e., $\mathcal{S}(\mathbf{r}',t)\approx\mathcal{I}_{11}\Phi(k_{11},t)\Psi(\mathbf{r}',k_{11})+\mathcal{I}_{21}\Phi(k_{21},t)\Psi(\mathbf{r}',k_{21})+\mathcal{I}_{12}\Phi(k_{12},t)\Psi(\mathbf{r}',k_{12})+\cdots$. The inset in Fig. 2 represents the quantity of particles absorbed by the surface of the system, i.e., $\mathcal{M}(\mathbf{r}',t)=1-\mathcal{S}(\mathbf{r}',t)$. By using Eq. (10), it is possible to find the mean first passage time by using equation $T(\mathbf{r}')=\int_0^\infty dt\mathcal{S}(\mathbf{r}',t)$ (see Fig. 3).

In this regard, notice that Fig. 3(a) evidences the influence of the spatial dependence present in the diffusion coefficient on the diffusion process which results in an anomalous diffusion.

Now, we extend the previous results for the limit $a\to\infty$. The solution for this case is formally the same of the previous case, but with the Green function given by

$$\mathcal{G}(\mathbf{r},\mathbf{r}',t)=-\frac{2}{2+\eta}\int_0^\infty dkk\sum_{m=1}^{\infty}\Phi(k,t)\tilde{\Psi}_m(\mathbf{r},k)\tilde{\Psi}_m(\mathbf{r}',k), \tag{12}$$

with

$$\tilde{\Psi}_m(\mathbf{r},k)=r^{\eta/2}\mathrm{J}_{\nu_m}\left(\frac{2k}{2+\eta}r^{1/2(2+\eta)}\right)\sin\left(\frac{m\pi}{\beta}\theta\right). \tag{13}$$

After some calculations, it is possible to simplify Eq. (12) to

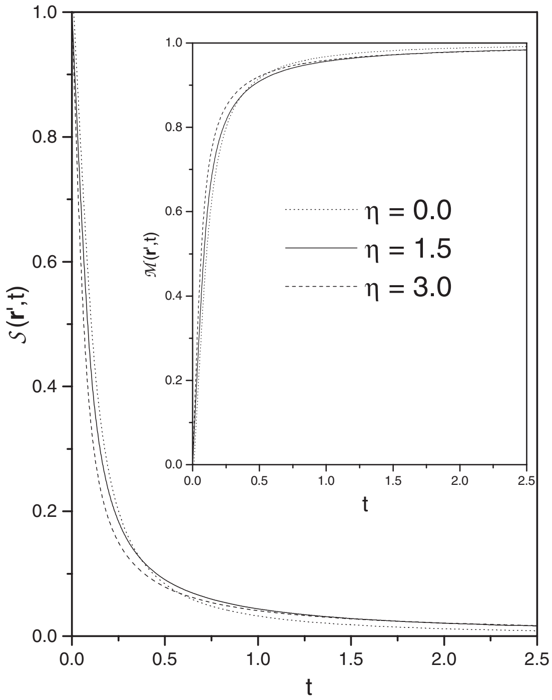

FIG. 4. Behavior of $\mathcal{S}(\mathbf{r}',t)$ versus t, which illustrates Eq. (15), for typical values of η by considering, for simplicity, $\mathcal{D}=1$, $\beta=2\pi/5$, and the initial condition $\rho(\mathbf{r},0)=1/r\delta(r-1)\delta(\theta-\pi/7)$. The inset figure is $\mathcal{M}(\mathbf{r}',t)$ versus t.

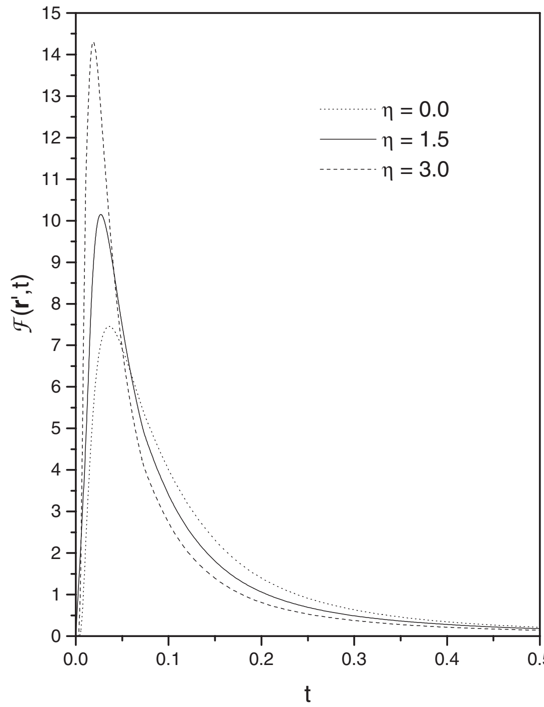

FIG. 5. Behavior of $\mathcal{F}(\mathbf{r}',t)$ versus t, which illustrates the first passage time distribution related to the process described by Eq. (14), for typical values of η by considering, for simplicity, $\mathcal{D}=1$, $\beta=\pi/3$, and the initial condition $\rho(\mathbf{r},0)=1/r\delta(r-1)\delta(\theta-\pi/5)$.

$$\mathcal{G}(\mathbf{r},\mathbf{r}',t)=-\frac{2(rr')^{\eta/2}}{(2+\eta)\beta\mathcal{D}t}e^{-r^{2+\eta}+r'^{2+\eta}/(2+\eta)^2\mathcal{D}t}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{\beta}\theta\right)\times\sin\left(\frac{m\pi}{\beta}\theta'\right)\mathrm{I}_{\nu_m}\left(\frac{2(rr')^{1/2(2+\eta)}}{(2+\eta)\mathcal{D}t}\right), \quad (14)$$

where $\mathrm{I}_\nu(x)$ is the Bessel function of modified argument. The survival probability related to the process described by Eq. (14), for the initial condition $\rho(\mathbf{r},0)=1/r\delta(r-r')\delta(\theta-\theta')$, is given by

$$\mathcal{S}(\mathbf{r}',t)=\frac{2\beta r'^{\eta/2}e^{-r'^{2+\eta}/(2+\eta)^2\mathcal{D}t}}{\pi[(2+\eta)^2\mathcal{D}t]^{\eta/2(2+\eta)}}\sum_{m=1}^{\infty}\frac{[1-(-1)^m]}{m\Gamma(1+\nu_m)}\sin\left(\frac{m\pi}{\beta}\theta'\right)\times\Gamma\left[\frac{4+\eta}{2(2+\eta)}+\frac{\nu_m}{2}\right]\left[\frac{4r'^{2+\eta}}{(2+\eta)^2\mathcal{D}t}\right]^{\nu_m/2}\times\Phi\left[\frac{\nu_m}{2}+\frac{4+\eta}{2(2+\eta)},\nu_m+1,\frac{r'^{2+\eta}}{(2+\eta)^2\mathcal{D}t}\right], \quad (15)$$

where $\Phi(a,b,x)$ is a hypergeometric function [51]. The asymptotic behavior of Eq. (15) for long times is $\mathcal{S}(t)\sim 1/t^{\eta/2(2+\eta)+\nu_1/2}$, which shows explicitly the dependence on the variable η. Note that the asymptotic behavior decays as a power law in time with characteristic exponent dependent on η and on the wedge opening angle. Figure 4 illustrates the survival probability related to this process which has a behavior similar to the previous result obtained from Eq. (10).

In Fig. 5, we illustrate the first passage time distribution obtained from Eq. (15) by using $\mathcal{F}(\mathbf{r}',t)=-\partial\mathcal{S}(\mathbf{r}',t)/\partial t$ [44], since the system is not limited on r.

Let us incorporate the external force $\bar{F}(r)=-kr\hat{r}+\mathcal{K}\hat{r}/r^{1+\eta}$ to the previous scenario and obtain the solution for Eq. (1), the survival probability and the first passage time distribution. Note that the external force considered here has the same parameter η as the diffusion coefficient. This choice was performed in order to avoid the cumbersome calculations which emerge by considering different parameters. Following the procedure employed in the previous cases, we also use the Green function approach to analyze the problem. After some calculations, it is possible to show that the Green function is now given by

$$\mathcal{G}(\mathbf{r},\mathbf{r}',t)=r^{\mathcal{K}/\mathcal{D}}e^{-kr^{2+\eta}/(2+\eta)\mathcal{D}}\sum_{n=0,m=1}^{\infty}\Phi(\lambda_{nm},t)\hat{\Psi}_{mn}(\mathbf{r})\hat{\Psi}_{mn}(\mathbf{r}'), \quad (16)$$

with $\bar{\alpha}_m=\sqrt{(2\pi m/\beta)^2+(\mathcal{K}/\mathcal{D}-\eta)^2}/(2+\eta)$,

$$\hat{\Psi}_{mn}(\mathbf{r})=r^{\xi}\mathrm{L}_n^{\bar{\alpha}_m}\left(\frac{kr^{2+\eta}}{(2+\eta)\mathcal{D}}\right)\sin\left(\frac{m\pi}{\beta}\theta\right), \quad (17)$$

where $\xi=(\eta-\mathcal{K}/\mathcal{D}+(2+\eta)\bar{\alpha}_m)/2$, $\mathrm{L}_n^{\alpha}(x)$ is the associated Laguerre polynomial [51], the time dependence is given by

$$\hat{\Phi}(\lambda_{nm},t)=-\frac{2k}{\beta\mathcal{D}}\frac{\Gamma(1+n)}{\Gamma(1+n+\bar{\alpha}_m)}\left(\frac{k}{2+\eta}\right)^{\bar{\alpha}_m}e^{-\lambda_{mn}t} \quad (18)$$

and $\lambda_{mn}=(2+\eta)nk+(k/2)(\eta-\mathcal{K}/\mathcal{D}+(2+\eta)\bar{\alpha}_m)$. We underline that the presence of the external force is not enough to assure a stationary solution. Similarly to what happened in the previous case, it is possible to simplify Eq. (16) to obtain

$$\mathcal{G}(\mathbf{r},\mathbf{r}',t) = -(rr')^{1/2(\eta+\mathcal{K}/\mathcal{D})}e^{-k/2(\eta-\mathcal{K}/\mathcal{D})t}\frac{2kr^{\mathcal{K}/\mathcal{D}}e^{-kr^{2+\eta}/(2+\eta)\mathcal{D}}}{\beta\mathcal{D}(1-e^{-(2+\eta)kt})}e^{-ke^{-(2+\eta)kt}(r^{2+\eta}+r'^{2+\eta})/(2+\eta)\mathcal{D}(1-e^{-(2+\eta)kt})}$$
$$\times\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{\beta}\theta\right)\sin\left(\frac{m\pi}{\beta}\theta'\right)\mathrm{I}_{\bar{\alpha}_m}\left(\frac{2k\Lambda(t)}{(2+\eta)\mathcal{D}}(rr')^{1/2(2+\eta)}\right), \quad (19)$$

where $\Lambda(t)=e^{-1/2(2+\eta)kt}/[1-e^{-(2+\eta)kt}]$. Therefore, the survival probability does not decay as a power law such as in the previous case due to the presence of external forces acting on the system. Notice that, in Fig. 6, for $\eta=0$ the particles remain in the system for more time than in the case $\eta>0$ for all time, in contrast to the situation analyzed before, which is characterized by the absence of external forces. It is worth mentioning that the case characterized by the absence of external force recovers results found in [43] when $\eta=0$. In particular, the lowest order corrections, in the absence of external force, for the case $\eta=0$, may be obtained by iterating the integral equation

$$\rho(\mathbf{r},t) = -\int_0^{\infty}dr'r'\int_0^{\beta}d\theta'\rho(\mathbf{r}',0)\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}',t) - \int_0^t dt'\int_0^{\infty}dr'r'\int_0^{\beta}d\theta'\alpha(\mathbf{r}',t')\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}',t-t') \quad (20)$$

where $\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}',t)=\mathcal{G}(\mathbf{r},\mathbf{r}',t)|_{\eta=0}$, $\alpha(\mathbf{r},t)=\mathcal{D}\nabla\cdot[\xi(r,\eta)\nabla\rho(\mathbf{r},t)]$ and $\xi(r,\eta)=r^{-\eta}-1\approx-\eta\ln r+(\eta^2\ln^2 r)/2+\mathcal{O}(\eta^3)$, with $0<\eta\ll 1$. In Fig. 7, we show the first passage time distribution related to Eq. (19) in order to illustrate the influence of the external forces on this quantity.

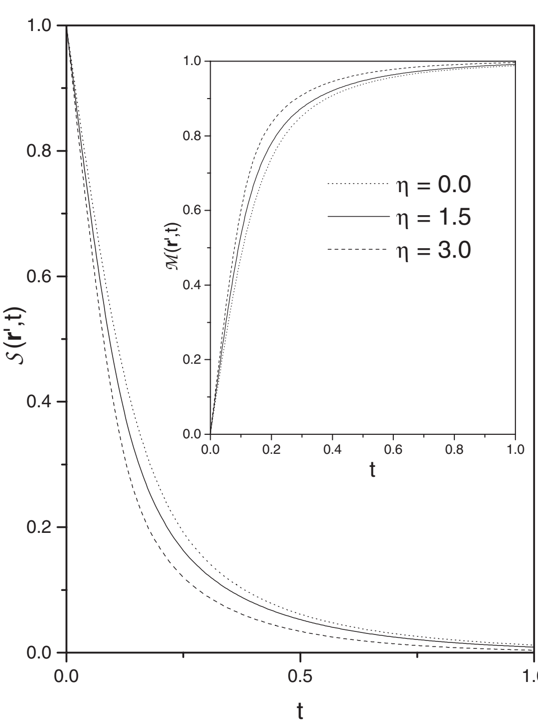

FIG. 6. Behavior of $\mathcal{S}(\mathbf{r}',t)$ versus t, which illustrates the diffusion process governed by Eq. (16), for typical values of η by considering, for simplicity, $\mathcal{D}=1$, $\beta=\pi/3$, $k=1$, $\mathcal{K}=1$, and the initial condition $\rho(r,0)=1/r\delta(r-1)\delta(\theta-\pi/5)$. The inset figure is $\mathcal{M}(\mathbf{r}',t)$ versus t.

III. DISCUSSIONS AND CONCLUSIONS

We have investigated the solutions of Eq. (1) by accomplishing several situations characterized by a spatially dependent diffusion coefficient and by the presence of external forces. The first situation analyzed refers to a limited wedge region. For this case, the survival probability presented an anomalous behavior for $\eta>0$. In fact, for small times, Fig. 2 shows that $\mathcal{S}(t)$ has its small value for $\eta=3$ (see the dashed and dotted lines) indicating that more particles are initially removed from the system for $\eta>0$ than for the usual case $\eta=0$. However, for long time, this behavior observed for $\mathcal{S}(t)$ changes and $\mathcal{S}(t)$ attains its small value for $\eta=0$. In this framework, we have also investigated the mean first passage

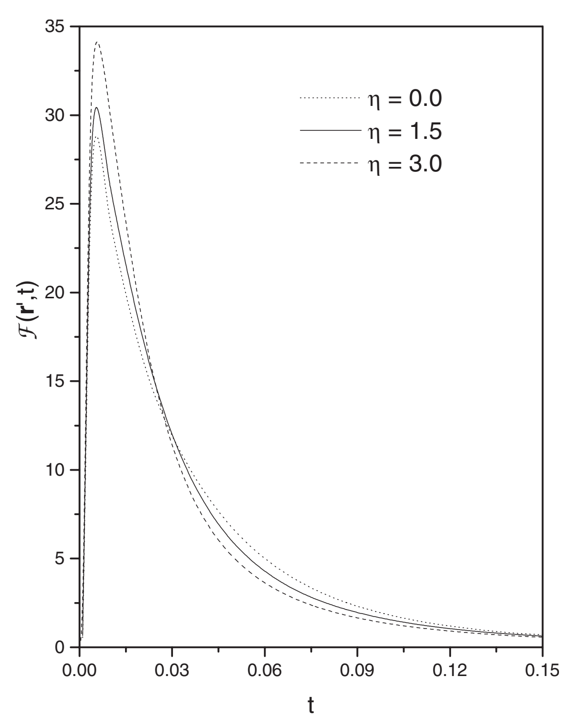

FIG. 7. Behavior of $\mathcal{F}(\mathbf{r}',t)$ versus t, which illustrates the first passage time distribution related to the process described by Eq. (19), for typical values of η by considering, for simplicity, $\mathcal{D}=1$, $\beta=\pi/5$, $k=1$, $\mathcal{K}=1$, and the initial condition $\rho(\mathbf{r},0)=1/r\delta(r-1)\delta(\theta-\pi/7)$. Similarly to what is shown in Fig. 5, the first time distributions have their maxima for small values of time, in agreement with the graphics presented for $\mathcal{S}(\mathbf{r}',t)$ in Fig. 6.

time which manifested a nonsymmetric behavior in r' due to the inhomogeneous characteristics of the diffusion coefficient. After that, we have extended these results by considering the limit $a \to \infty$ and by incorporating an external force to the problem. Similarly to the previous results, we have also obtained an anomalous behavior for the survival probability and for the first passage time distribution, when $\eta > 0$. In this direction, results found in [43,44] have been extended by incorporating external forces and a spatial dependence in the diffusion coefficient. In this manner, the results obtained here should be useful to investigate anomalous diffusion in a wedge region.

ACKNOWLEDGMENTS

We thank CNPq/INCT-SC and Fundação Araucária (Brazilian agencies) for the financial support.

[1] A. Ott, J. P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990).
[2] J. P. Bouchaud, A. Ott, D. Langevin, and W. Urbach, J. Phys. II **1**, 1465 (1991).
[3] C.-K. Peng, J. Mietus, J. M. Hausdorff, S. Havlin, H. E. Stanley, and A. L. Goldberger, Phys. Rev. Lett. **70**, 1343 (1993).
[4] T. H. Solomon, E. R. Weeks, and H. L. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).
[5] F. Bardou, J. P. Bouchaud, O. Emile, A. Aspect, and C. Cohen-Tannoudji, Phys. Rev. Lett. **72**, 203 (1994).
[6] G. M. Zaslavsky, D. Stevens, and H. Weitzner, Phys. Rev. E **48**, 1683 (1993).
[7] G. M. Zaslavsky, Physica D **76**, 110 (1994).
[8] G. M. Zaslavsky, Chaos **4**, 25 (1994).
[9] J. Klafter and G. Zumofen, Phys. Rev. E **49**, 4873 (1994).
[10] H. Spohn, J. Phys. I **3**, 69 (1993).
[11] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[12] R. Metzler and J. Klafter, J. Phys. A **37**, R161 (2004).
[13] R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000).
[14] L. S. Lucena, L. R. da Silva, L. R. Evangelista, M. K. Lenzi, R. Rossato, and E. K. Lenzi, Chem. Phys. **344**, 90 (2008).
[15] E. K. Lenzi, M. K. Lenzi, R. Rossato, L. R. Evangelista, and L. R. Silva, Phys. Lett. A **372**, 6121 (2008).
[16] R. Rossato, M. K. Lenzi, L. R. Evangelista, and E. K. Lenzi, Phys. Rev. E **76**, 032102 (2007).
[17] L. C. Malacarne, R. S. Mendes, E. K. Lenzi, and M. K. Lenzi, Phys. Rev. E **74**, 042101 (2006).
[18] W. R. Schneider and W. Wyss, J. Math. Phys. **30**, 134 (1989).
[19] E. K. Lenzi, R. S. Mendes, K. S. Fa, L. C. Malacarne, and L. R. Silva, J. Math. Phys. **44**, 2179 (2003).
[20] E. K. Lenzi, R. S. Mendes, J. S. Andrade, L. R. da Silva, and L. S. Lucena, Phys. Rev. E **71**, 052101 (2005).
[21] P. C. Assis, R. P. de Souza, P. C. da Silva, L. R. da Silva, L. S. Lucena, and E. K. Lenzi, Phys. Rev. E **73**, 032101 (2006).
[22] T. A. M. Langlands, Physica A **367**, 136 (2006).
[23] R. Metzler and T. F. Nonnenmacher, Chem. Phys. **284**, 67 (2002).
[24] A. R. Plastino and A. Plastino, Physica A **222**, 347 (1995); C. Giordano, A. R. Plastino, M. Casas, and A. Plastino, Eur. Phys. J. B **22**, 361 (2001).
[25] C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996).
[26] E. Arvedson, M. Wilkinson, B. Mehlig, and K. Nakamura, Phys. Rev. Lett. **96**, 030601 (2006).
[27] F. Mota-Furtado and P. F. O'Mahony, Phys. Rev. E **75**, 041102 (2007).
[28] V. Bezuglyy, B. Mehlig, M. Wilkinson, K. Nakamura, and E. Arvedson, J. Math. Phys. **47**, 073301 (2006).
[29] F. Mota-Furtado and P. F. O'Mahony, Phys. Rev. A **74**, 044102 (2006).
[30] J. Bisquert, Phys. Rev. E **72**, 011109 (2005).
[31] J. L. A. Dubbeldam, A. Milchev, V. G. Rostiashvili, and T. A. Vilgis, Phys. Rev. E **76**, 010801(R) (2007).
[32] R. Metzler, E. Barkai, and J. Klafter, Physica A **266**, 343 (1999).
[33] R. Metzler, G. Glöckle, and T. F. Nonnenmacher, Physica A **211**, 13 (1994).
[34] B. O'Shaughnessy and I. Procaccia, Phys. Rev. Lett. **54**, 455 (1985).
[35] P. Demontis and G. B. Suffritti, Phys. Rev. E **74**, 051112 (2006).
[36] U. Achatz, J. Steinacker, and R. Schlickeiser, Astron. Astrophys. **250**, 266 (1991).
[37] O. Stawicki, J. Geophys. Res. **109**, A04105 (2004).
[38] G. Boffetta and I. M. Sokolov, Phys. Rev. Lett. **88**, 094501 (2002).
[39] S. Chaudhury and B. J. Cherayila, J. Chem. Phys. **127**, 145103 (2007).
[40] M. O. Vlad, Chaos, Solitons Fractals **4**, 191 (1994).
[41] H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1984).
[42] C. W. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Sciences* (Springer Series in Synergetics, New York, 1996).
[43] Diandrew Lexter L. Dy and J. P. Esguerra, Phys. Rev. E **78**, 062101 (2008).
[44] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, England, 2001).
[45] H. K. Moffatt and B. R. Duffy, J. Fluid Mech. **96**, 299 (1980).
[46] D. A. Kessler, Z. Olami, J. Oz, I. Procaccia, E. Somfai, and L. M. Sander, Phys. Rev. E **57**, 6913 (1998).
[47] S. Redner, Phys. Rev. E **56**, 4967 (1997).
[48] A. A. Vedenov, Rev. Plasma Phys. **3**, 229 (1967).
[49] M. P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[50] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).
[51] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 2007).